Multiplication

$$12 \times 7 = 84$$

factor × factor = product

Concrete Pictorial Abstract

referred to as CPA

The concrete-pictorial-abstract approach, based on research by psychologist Jerome Bruner, suggests that there are three steps (or representations) necessary for pupils to develop understanding of a concept. Reinforcement is achieved by going back and forth between these representations.

Concrete representation

The enactive stage - a student is first introduced to an idea or a skill by acting it out with real objects. In division, for example, this might be done by separating apples into groups of red ones and green ones or by sharing 12 biscuits amongst 6 children. This is a 'hands on' component using real objects and it is the foundation for conceptual understanding

Pictorial representation

The iconic stage - a student has sufficiently understood the hands-on experiences performed and can now relate them to representations, such as a diagram or picture of the problem. In the case of a division exercise this could be the action of circling objects

Abstract representation

The symbolic stage - a student is now capable of representing problems by using mathematical notation, for example: 12 ÷ 2 = 6 This is the ultimate mode, for it "is clearly the most mysterious of the three."

Use a range of 'concrete objects' / 'tools to think with'

Begin with real life situations so that children use objects/'tools to think with' to represent the situation. Initially these need to be the objects in the question. Later the objects can be representative.

Using real life questions and modelling using 'concrete objects' as 'tools to think with' will ensure that children develop their ***visualisation skills*** (mentally thinking in your head).

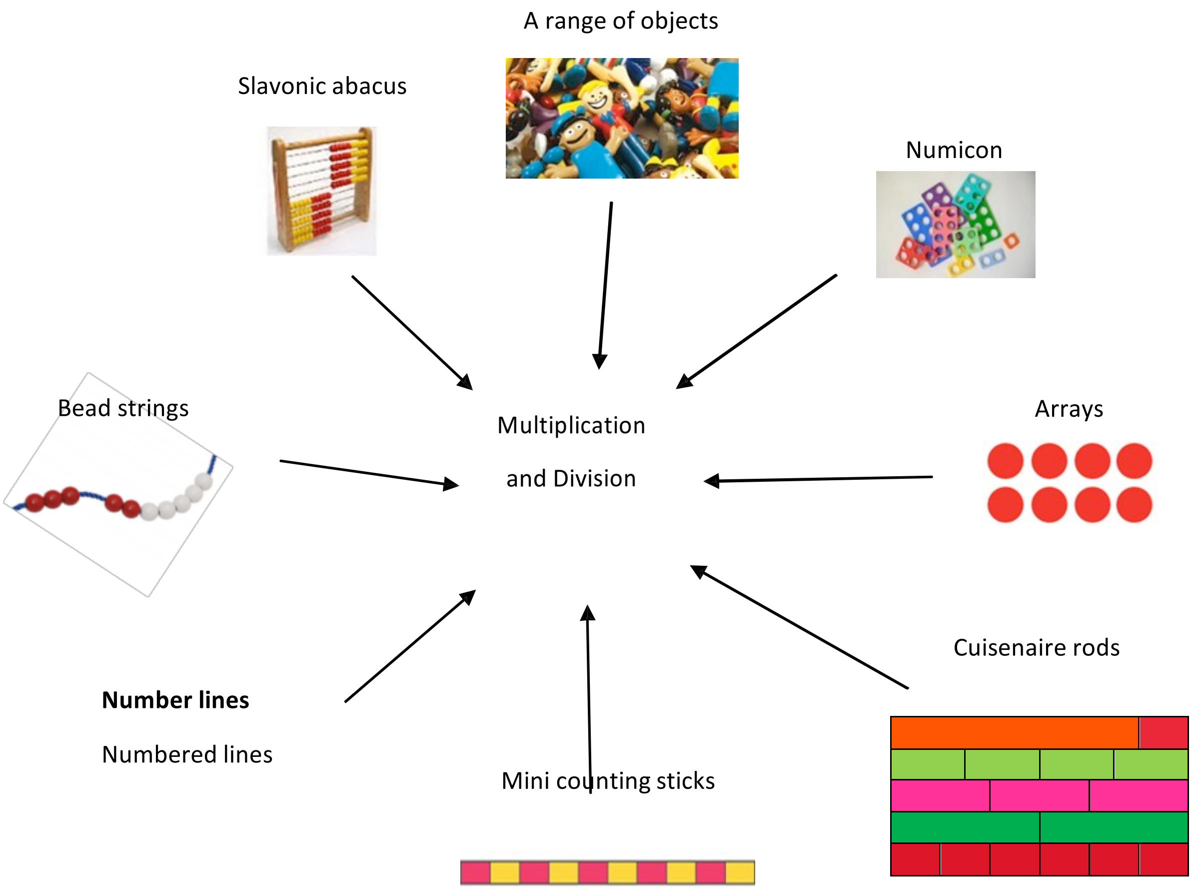
Visualisation is a key intellectual competency—it supports pupil's skills in 'seeing'/visualising what a questions is asking of them. It is very important to develop visualisation so that pupils have this skill as they move from the concrete, to pictorial and then to the abstract.

The development of visualisation skills requires a focussed approach. Visualisation needs to be discussed with the children through the use of questions i.e. What are you picturing in your head?

The importance of visualisation, in terms of how visualisation will support their learning now and in the future, needs to be shared with the pupils.

If, when solving a problem, the focus is on the answer only, then the pupils may not be exposed to the opportunity to develop their ability to visualise, thus restricting their ability to move from concrete representation (enactive) to pictorial representation (iconic) and then to abstract representation (symbolic).

‘Tools to think with’



Learning

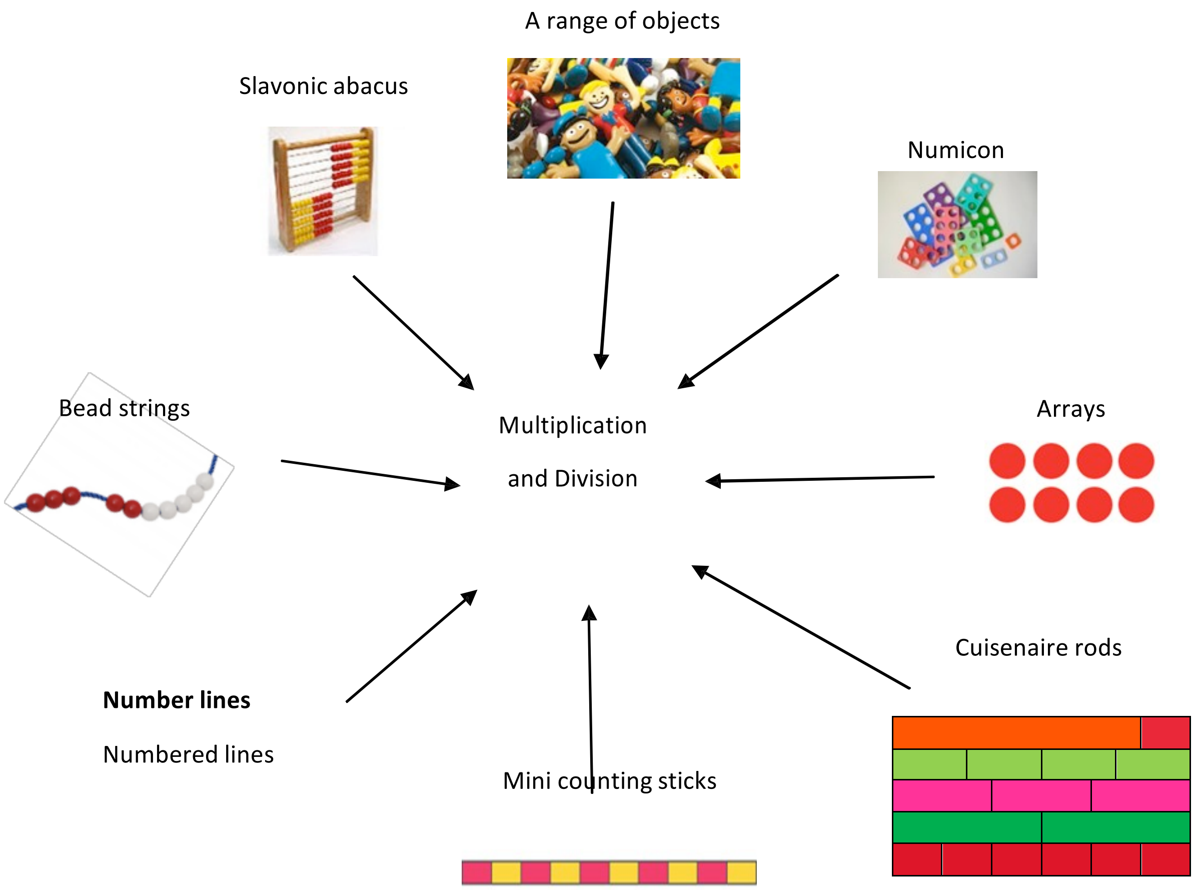
Learning and teaching are interrelated: one does not occur without the other. Genuine teaching is not linear. It is messy, arrived at by many paths, and characterised by different-sized steps and shifts in direction. Genuine teaching is directed towards landmarks and horizons…. As we learn we construct. (Fosnot and Dolk 2001, p.30)

Word problems V context problems

Word problems are often designed with little context they are usually nothing more than superficial, camouflaged attempts to get children to do the procedures teachers want them to do.

Context problems are connected as closely as possible to children’s lives, rather than to ‘school mathematics’(Fosnot and Dolk 2001p. 26).

‘Tools to think with’



Counting in multiples of (step counting) Use the Concrete - Pictorial - Abstract approach

When counting in multiples of/step counting - it is important to ensure that the counting is modelled using real-life objects and concrete representations (cardinal) for example Numicon as well as counting sticks and a number line (ordinal).

Year 1: Count in twos, fives, and tens	Use socks, shoes, hands, gloves, Numicon...
Year 2: Count in steps of 2, 3 and 5 from 0, and in tens	Use Numicon, counting sticks and number lines
Year 3: Count from 0 in multiples of 4, 8, 50 and 100	Use Numicon, counting sticks and number lines
Year 4: Count in multiples of 6, 7, 9 and 1000	Use Numicon, counting sticks and number lines
Year 5: Count forwards or backwards in steps of powers	Use counting sticks and number lines

Big ideas in multiplication

Unitising

The importance of Unitising

"When children are attempting to understand "how many" (how many cakes there are in the box, for example), initially children use counting. They tag and count each object once and once only.

But counting in ones is not multiplication.

The understanding that four can simultaneously be one, one row, or one bag of four cakes - is the big idea of **unitising**.

Many researchers have described -

" how a number must be treated differently when it is unitised, a difficult idea for children".

Prior to constructing this idea, number is used to represent single units - six represent six cookies.

To understand that this group can be counted simultaneously as one, requires a higher-order treatment of number in which groups are counted as well as the objects in the group."

When children can **unitise** they start to use the language of unitising, for example *"five fours"*

This language demonstrates the unitising of units.

The child is counting groups not objects.

"The whole is thus seen as a number of groups of a number of objects - for example four groups of six, or 4 x 6. The parts together become the new whole, and the parts (the groups) and the whole can be considered simultaneously.

The relationship of these to the whole explain the reciprocal relationship between division and multiplication.

Because we know the parts (the number of objects in each group and the number of groups), we can figure out the whole.

If we know the whole and one part (the number of groups, say) we can figure out the other part (the number of objects in the group).

Unitising is also a central organising idea in mathematics because it underlies the understanding of place value: ten objects are one ten."

Multiplication - a binary operation

"Addition and subtraction can be thought of as the joining of sets, multiplication is about replication.

Addition and subtraction are **unary operations** with each input representing the same kind of element – 3 blocks added to 4 blocks or 3 oranges added to 4 oranges.

However, we need to view multiplication as a **binary operation** with two distinctive inputs (Anghileri, 2000). The first input represents the size of a set (say the number of oranges in a particular set) and the second represents the number of replications of that set (how many sets of oranges). In this way, the two numbers represent distinct elements of the multiplication process".

Barmby, P. and Harries, T. and Higgins, S. and Suggate, J. (2009) 'The array representation and primary children's understanding and reasoning in multiplication.', Educational studies in mathematics., 70 (3). pp.217-241.

Unary representations

Binary representations

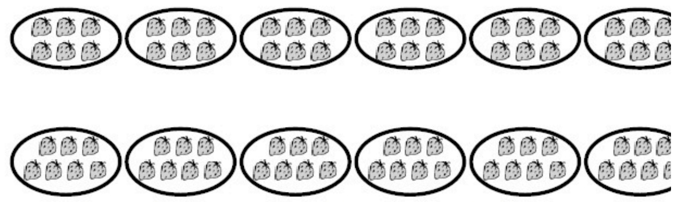


Figure 2: 'Groups of' representation of multiplication, showing 6 × 7 and 7 × 6

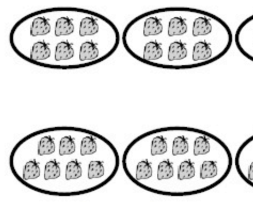


Figure 2: 'Groups of' representation

Figure 4: Array representation for 6 x7 or 7 x 6

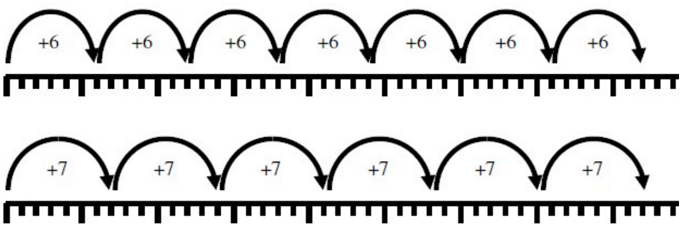


Figure 3: Number line representation of multiplication, showing 6 × 7 and 7 × 6

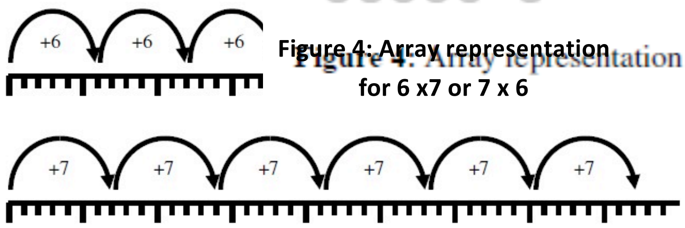


Figure 3: Number line representation of multiplication, showing 6 × 7 and 7 × 6

"Figures 2 and 3 show representations that encourage **unary** thinking and also encourage a repeated addition method of calculation – a method that whilst it may work becomes increasingly inefficient as the value of the numbers increase. These representations, which encourage repeated addition, are also problematic when the multiplication involves two rational numbers (fractions, negative numbers).

Both representations illustrate the idea of equal groups, and the number line provides help with the calculation.

Neither representation however illustrates the two important aspects of multiplication, [ ]namely commutativity and the distributive characteristic.

For example, when the numbers are swapped in the above diagrams, the representations will look quite different. It is not immediately obvious why the commutative law should apply. [ ]The array representation, shown below in Figure 4, encourages pupils to develop their thinking about multiplication as a **binary** operation with rows and columns representing the two inputs".

Barmby, P. and Harries, T. and Higgins, S. and Suggate, J. (2009) 'The array representation and primary children's understanding and reasoning in multiplication.', Educational studies in mathematics., 70 (3). pp.217-241.

Understanding arrays as a big idea

Michael Battista et al 1998, identified four stages that children go through as they develop the ability coordinate rows and columns.

Initial stage: students structure arrays as one-dimensional paths.

They draw or fill a 3 x 6 array in an unidirectional way - perhaps by filling/drawing the borders first. Therefore they are not noticing the structure of rows and columns and they are not using the language of **unitising,** for example, *five fours*

Second stage: Students structure one of the dimensions (rows or columns), but not both.

Children at this stage talk about repeated addition in the rows or columns but are unable to consider both simultaneously.

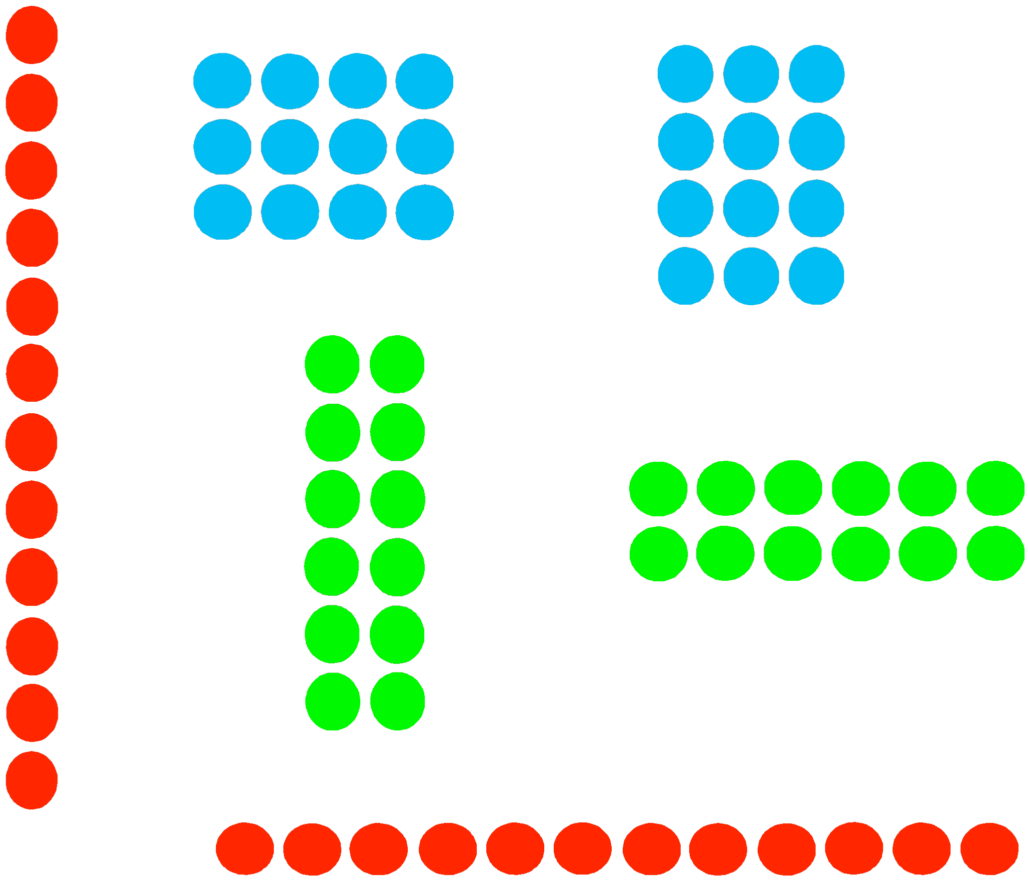
Third stage: Students become able to use the square units as indicators of how many rows and how many columns, but they still struggle to understand that one square can simultaneously represent a column and a row.

Arrays

Here are 12 counters.

Arrange them in equal rows.

Try different ways



Investigate arrays

Explore making arrays with different numbers

Here are 6, 9, 12, 13, 15, 16, 17, 18, 20, 24, 25, 36… counters

Find different ways to arrange them in equal rows.

What do you notice? What's the same? What's different?

Annotate your arrays - for example '3 fours' or 3 rows of 4, 3 lots of 4, 3 x 4

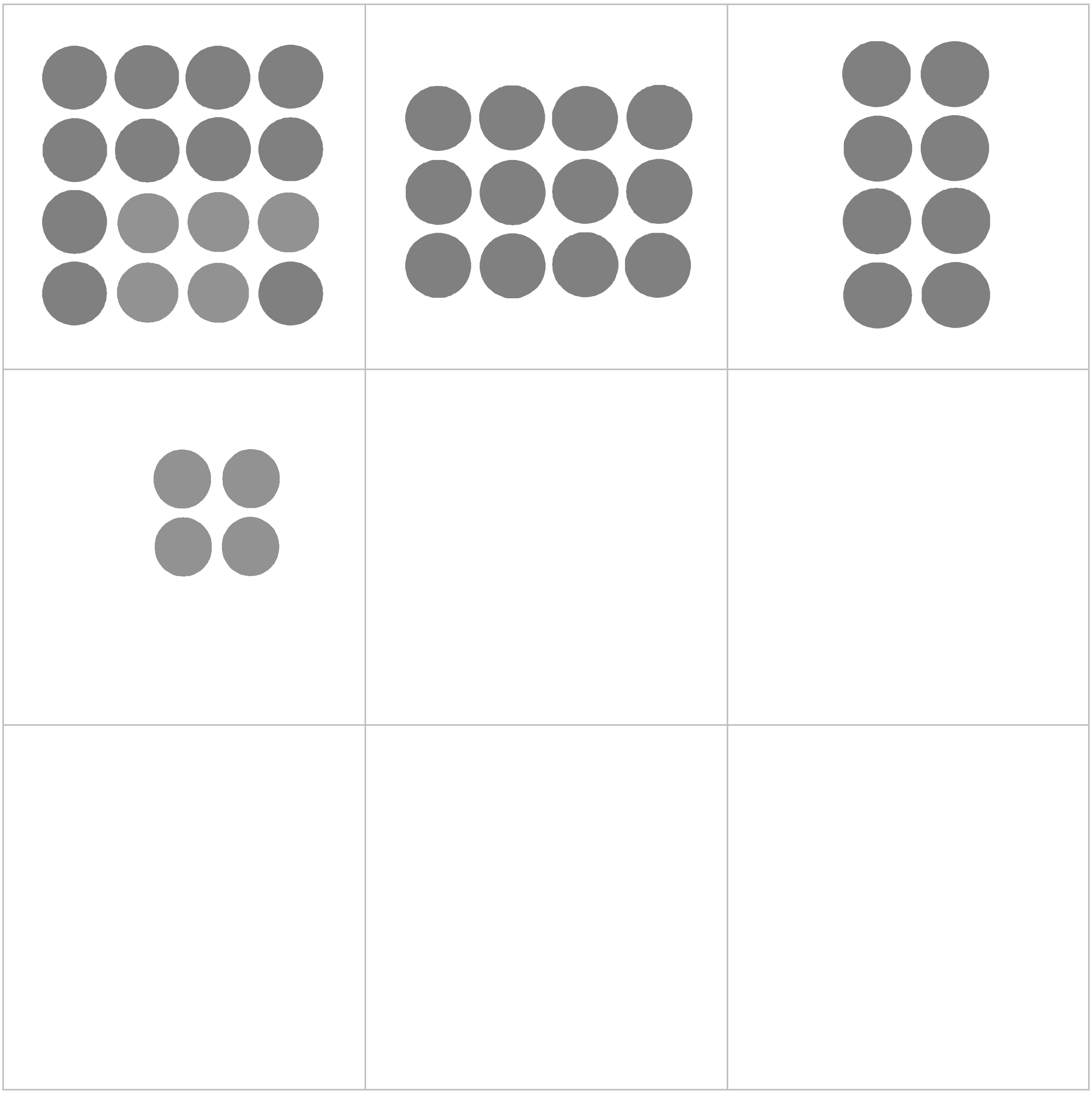
Array cards

Give children lots of opportunities to make and explore arrays: through making them with counters, by printing with circular objects or by pressing circular objects into play-dough.

Use the array cards below as flash cards. Ensure that the children can already subitise.

Let the children see the card for a short time and ask then to visualise the pattern of dots and say how many. Play snap. Play pairs.

To begin with children will need to count the number of dots but with practise they will learn to recognise the pattern of dots.



Representing multiplication situations as arrays

Match a situation (a context) to an array - a situation told verbally

Match a situation to an array and to a calculation

Match an array to a calculation and vice versa

Make links between arrays (see division policy)

Create or draw an array for a situation

Create or draw an array for a calculation

Draw an array for a situation or calculation and find how many by partitioning - (children should not be counting in ones—they should be encouraged to skip count or use the distributive property - partitioning)

Make links between different arrays - seeing one array in another and talk about and record the multiplicative relationships

Make links between area and perimeter and between volume and surface area - "how many different –sized boxes can you make to hold thirty-six Christmas ornaments and how much cardboard would be needed for each one. Making links to packaging (D&T). Three-dimensional arrays allow children to investigate the associative property for multiplication in a context that makes sense to them".

(Fosnot and Dolk 2001 p.45)

Ensure that children experience a range of multiplication situations that involve:

Equal groups - i.e. cookies on plates

Arrays

Constant rate -(money, speed, measures)

Area

Combination problem - find all possibilities i.e. 4 ice-cream flavours with 3 different toppings or clown hat problems

Scaling - find a ribbon that is 4 times as long as this one

Big ideas in multiplication

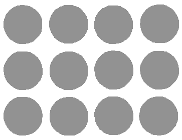
The Commutative Property

The Distributive Property

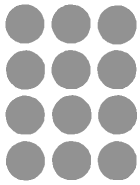
The Associative Property

It is important that these properties of multiplication are seen (noticed) and explored in two-dimensional arrays (commutative property) or with three–dimensional boxes - the (associative property).

The Commutative property



4 x 3 = 3 x 4



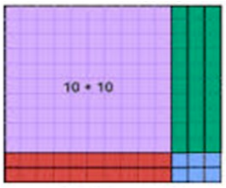
The Distributive property

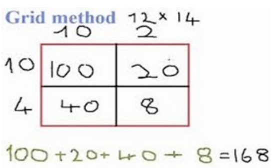
Realising that 9 x 5 can be solved by adding 5 x 5 and 4 x 5 or any combination of groups of five that add up to nine groups (for example 6 x 5 and 3 x 5) - this involves understanding about the structure of the part/whole relationship involved.

When using the distributive property , learners have to think about how to decompose the whole in groups.

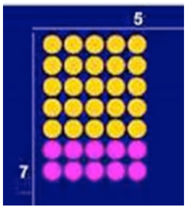
The distributive property is also a central organising idea in mathematics; it is the basis for the multiplication algorithm with whole numbers.

12 x 13 = (2 x 3) + (2 x 10) + (10 x 3) + (10 x 10)





7 x 5 = (5 x 5) + (2 x 5)

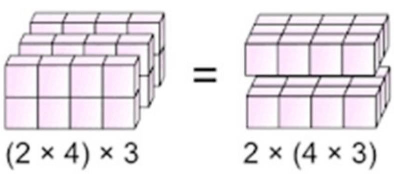


And in algebra (x + 1) (a + 4) = x² + 4x + x + 4

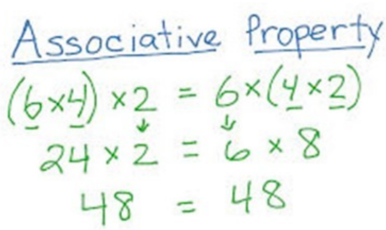
×	x	$+1$
x	x^2	$+1x$
$+4$	$+4x$	$+4$

Also that 7 x 8 = (8 x 8) - 8

The Associative property



(2 x 3) x 4 = 2 x (3 x 4)



Fact boxes

Children create fact boxes relevant to their understanding and to develop the range of facts they know and can use fluently.

In Year 2 children are learning the multiplication and division facts for the 2, 5 and 10 multiplication tables and are developing their fluency in applying these. They are also learning to count in steps of 2, 3 and 5 from 0 and in 10s from any number

In Year 3 children are learning the multiplication and division facts for the 3, 4 and 8 multiplication tables - therefore they should be fluent with the 2, 5 and 10 multiplication tables.

In Year 4 children are learning the multiplication and division facts for the 6, 7 and 9, 11 and 12 multiplication table - therefore they should be fluent with the 2, 3, 4, 5, 8 and 10 multiplication tables

$1 \longrightarrow 4$

$2 \longrightarrow 8$

$4 \longrightarrow 16$

$10 \longrightarrow 40$

$5 \longrightarrow 20$

$20 \longrightarrow 80$

Create and begin to explore your first fact box

Year 2 example:

Create a fact box with a group of children

Use a multiplication table that the children know fluently.

Create the fact box with the children, using this format.

Write the facts one at a time.

Children are usually happy with one lot of 5 and two lots of 5 but when the teacher writes 4 lots of 5 next, they often query this and ask why it is not three lots that go next. At this point it is important to talk to the children about the difference between what they have already experienced, that is 'times tables' which list all the multiples of 5 and a fact box which lists some of the multiples of 5.

Ask the children if it is possible to use the fact box to calculate 3 lots of 5 (children will know what 3 lots of 5 are but, we are asking them to identify how they can derive this fact from the information in the fact box). This can cause 'cognitive conflict' for children who have learnt the multiplication facts for 5 by rote. Here we are asking children to apply what they know in a more complex way. We are asking them to notice a connection - to notice that 2 lots of 5 added to 1 lot of 5 is equal to 3 lots of 5. Representing this as a array would be useful. However, if children cannot, fairly quickly, make this connection, then they are not ready to explore fact boxes.

Children will know the product of 4 lots of 5 but, we also ask them how they might use another fact in the box - double 2 lots if 5.

By this stage the children will have taken on board that a fact box includes some of the multiples of 5 and therefore will be happy that the next two multiples are 10 lots of 5 and 5 lots of 5. Again it is important to ask the children to notice connections between these two facts and other facts in the fact box.

Ask children to write out this fact box for themselves - use A3 paper and felt pens.

Ask the children to use the fact box to find the product of 7 lots of 5.

Once children are confident, ask them to work independently to find 9 lots of 5, 11 lots of 5, 13 lots of 5, 19 lots of 5 …

$1 \longrightarrow 5$

$2 \longrightarrow 10$

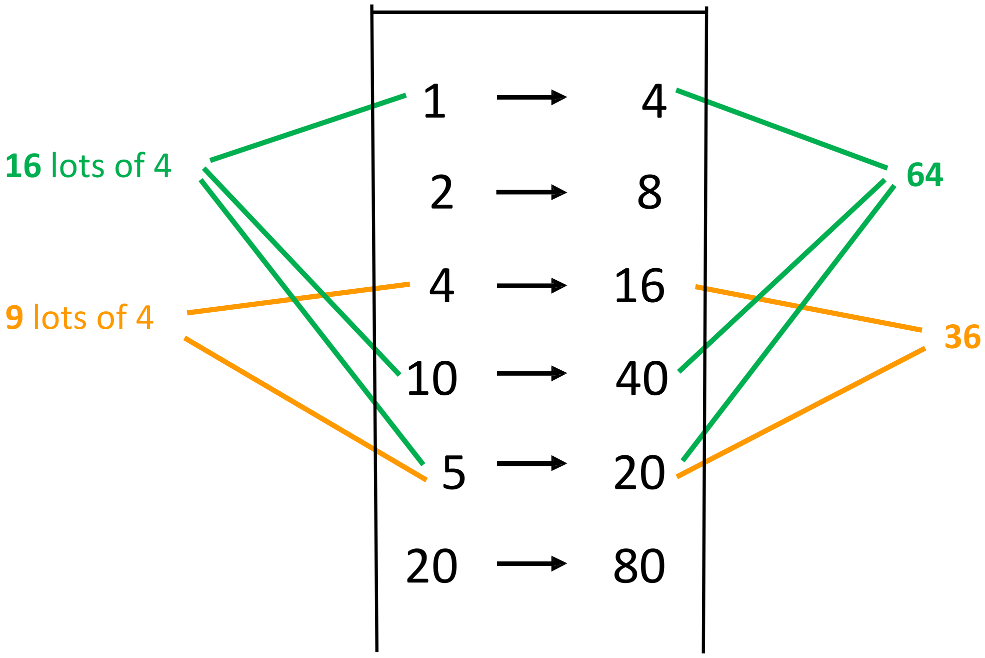
$4 \longrightarrow 20$

$10 \longrightarrow 50$

$5 \longrightarrow 25$

$20 \longrightarrow 100$

Explore multiplication using a fact box



Children are learning that **9 lots of 4** are equal to 5 lots of 4 plus 4 lots of 4

Also ask children to represent 9 lots of 4 (4 x 9, 9 x 4) with counters or by drawing an array. Arrange the counters to show that 5 lots of 4, plus 4 lots of 4 is equal to 9 lots of 4.

Use this fact box to calculate:

3 lots of 4, 3 x 4

7 lots of 4, 7 x 4

9 lots of 4, 9 x 4

Annotate your fact box

Ensure that colour is used mathematically

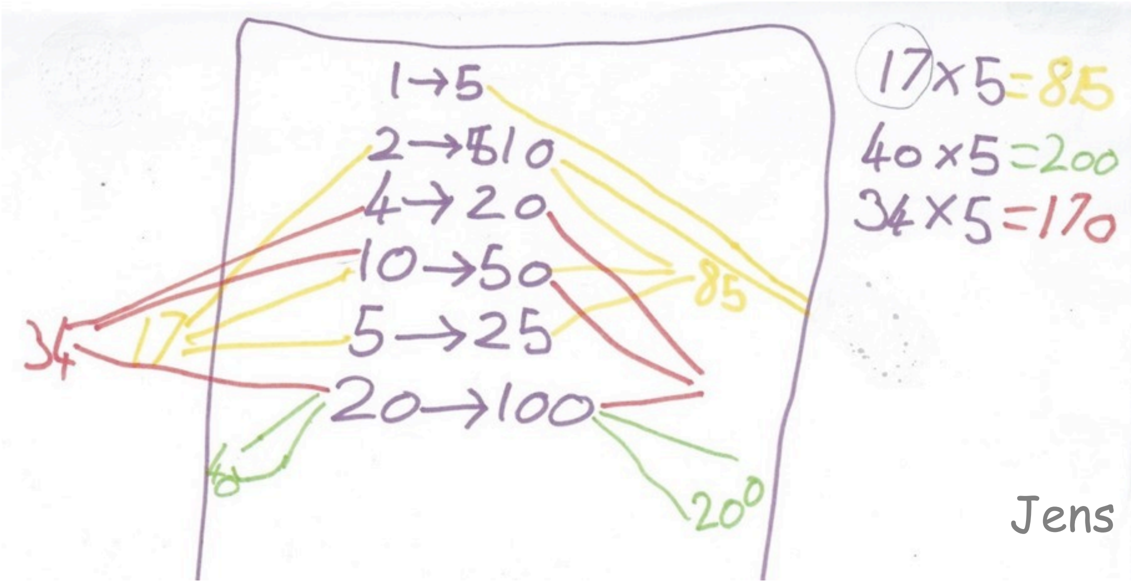
On the next page is an example of Jens's work.
Notice his effective use of colour.

How could we express how we calculated 16 x 4
16 x 4 = (10 x 4) + (5 x 4) + (1 x 4)
The distributive property (law)

Making connections

Here is an example of Jens's work - Year 2

Notice his effective use of colour.

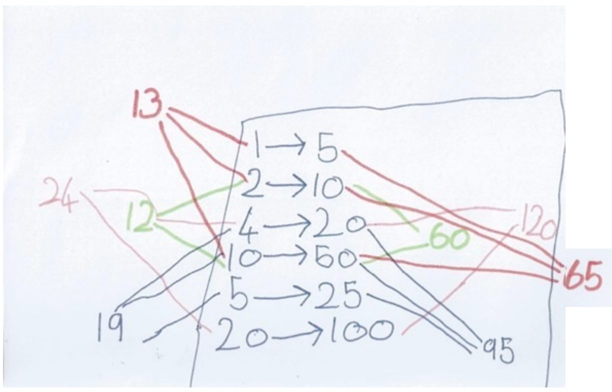


Spend time exploring fact boxes, developing children's 'number sense' and making the links to known facts (times tables), prior to calculating the answers to questions.

What do you notice?

What else can you calculate using your fact box?

What is the next fact you would add to your fact box? Why?



Make connections between multiplication and division

How many fives are there in 65?

Jens combines the products to make a total of 65 and then combines the number of fives to answer this question.

The recording is clear because of Jens's use of colour

Comparing fact boxes - making connections

What do you notice?

1	→	7	1	→	17
2	→	14	2	→	34
4	→	28	4	→	68
10	→	70	10	→	170
5	→	35	5	→	85
20	→	140	20	→	340

Can you calculate the product of 17 x 6, if you know the product of 7 x 6?

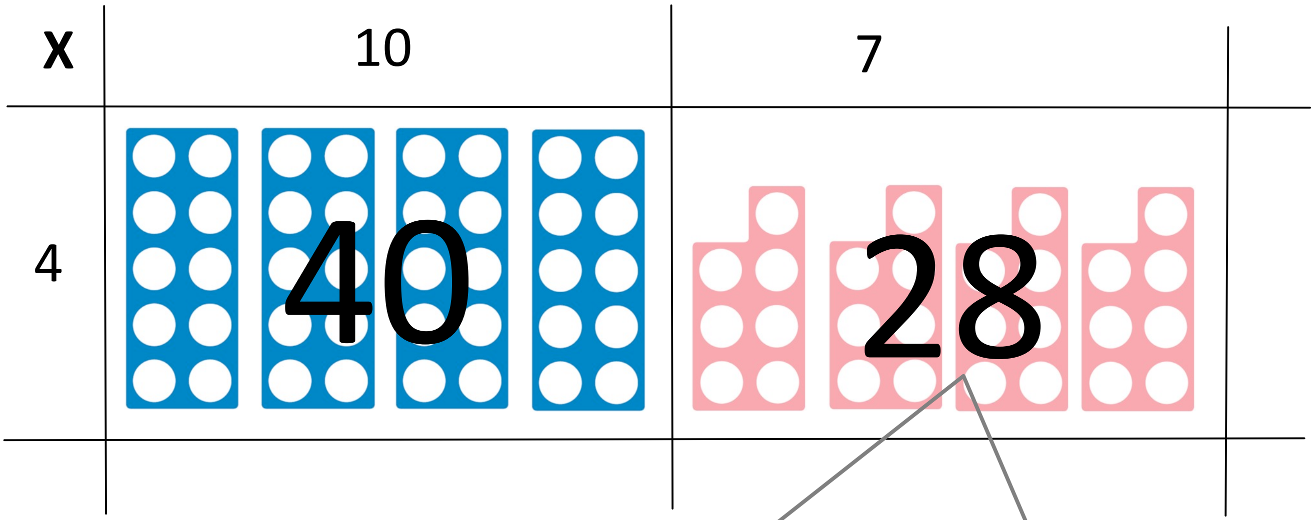
Why is the product of 17 x 4 forty more than the product of 7 x 4?

17 x 4 is 68. How can you use this fact to calculate the product of 27 x 4?

Multiplication - using Numicon to model the grid method

Applying mental calculation fluency

17 x 4



Children know this is 28 because:

They put the Numicon plates/shapes together and work out the total using the tens without counting

they are fluent at doubling - doubling 7 to get 14 and then doubling 14

they know their multiplication facts for the 7 times table (*year 4)*

Children combine the partial totals mentally to find the product of 17 x 4.

Later children might record this as

17 x 4

10 x 4 + 7 x 4

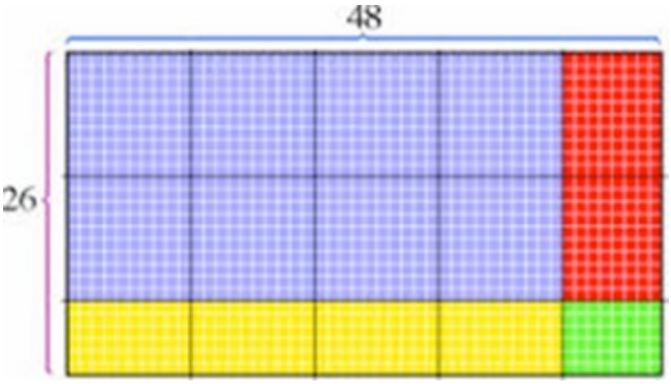
40 + 28 = 68

Multiplication - an array representation

Area model

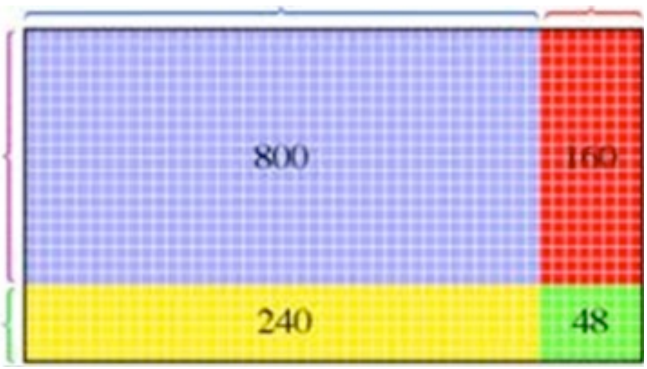
Area model - Partial products

Annotate this array to show how the **distributive property** has been used and to show the **partial products**



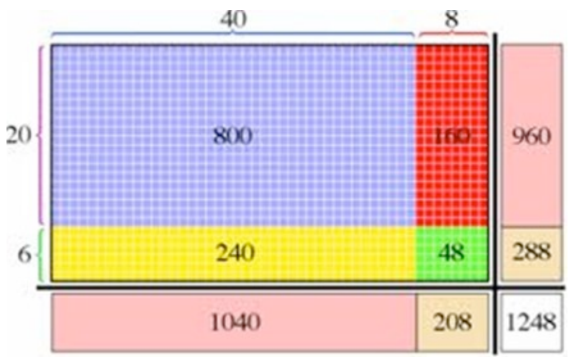
Annotate this array to show how the **distributive property** has been used and to show the **partial products** of the rows or columns.

Then find the **product** of 48 x 26



How will you add the partial products (the rows or the columns) to find the product of 48 x 26?

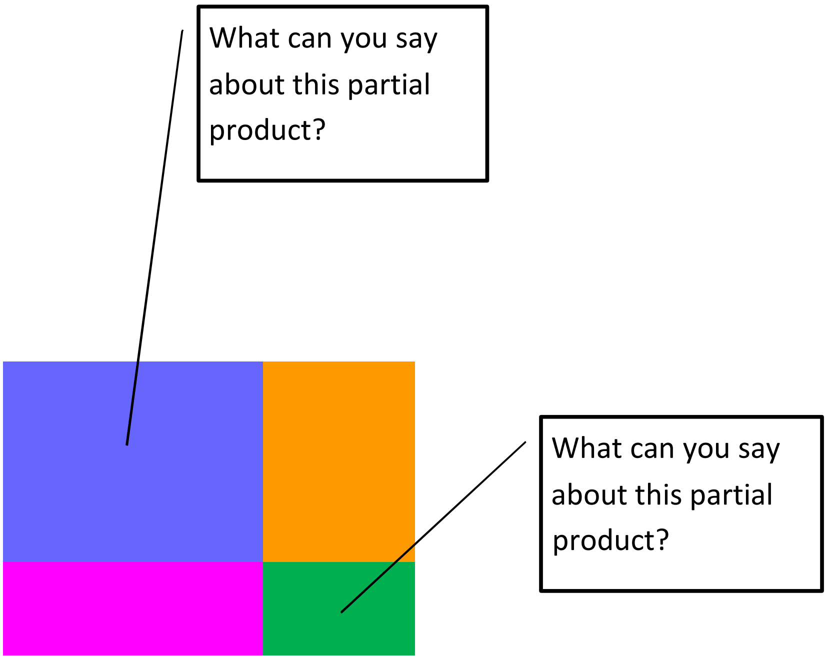
Why did you make that decision?

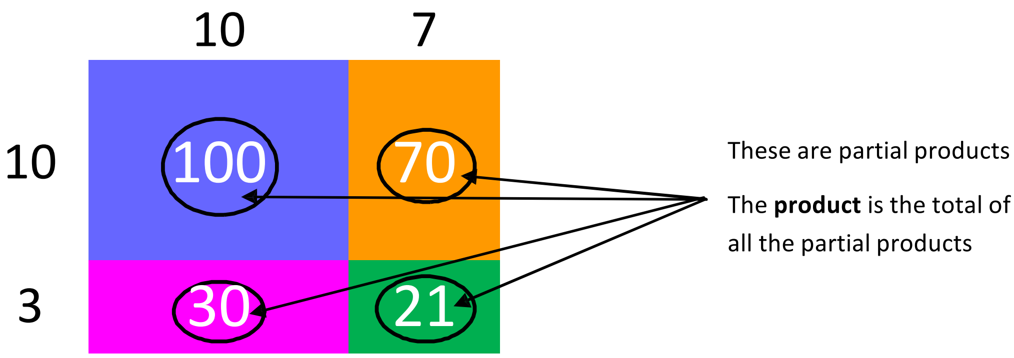


Multiplication - a geometrical representation

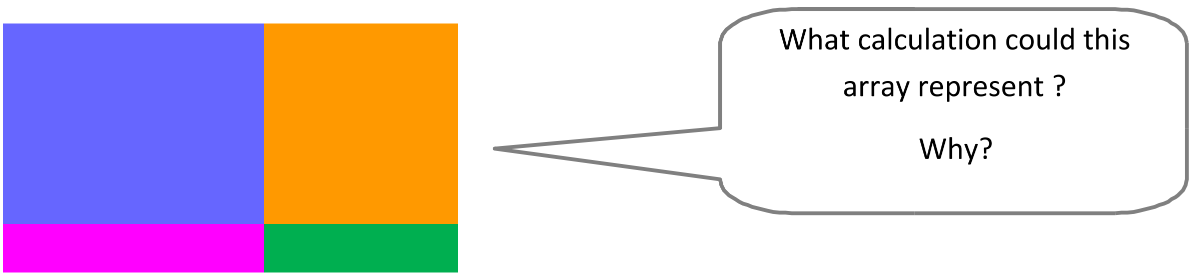
Area model

Area model - Partial products



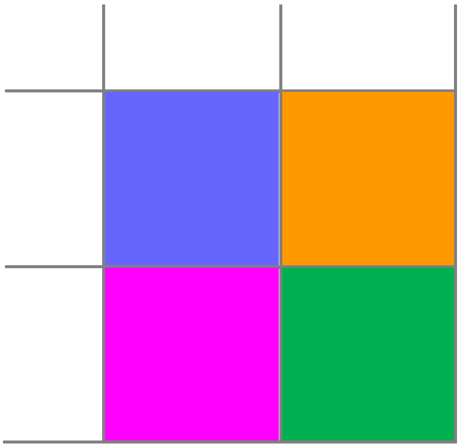


x



Multiplication—a geometrical representation

The size of each of the 'boxes' in the grid is the same

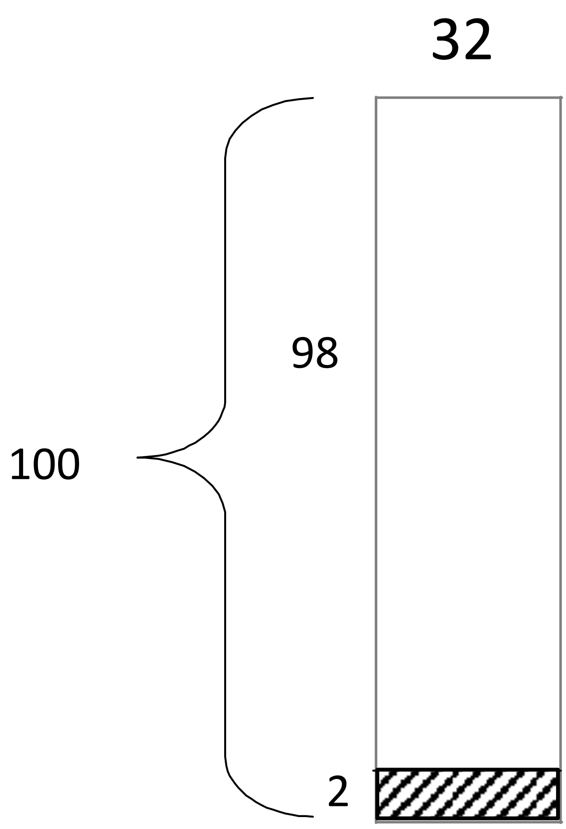


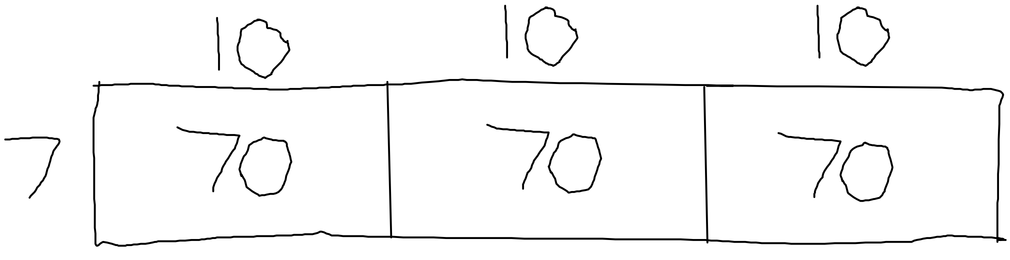
×	10	4
10		
6		

Open arrays

"A powerful tool to think with"

98 x 32 = (100 x 32) - (2 x 32) = 3200 - 64

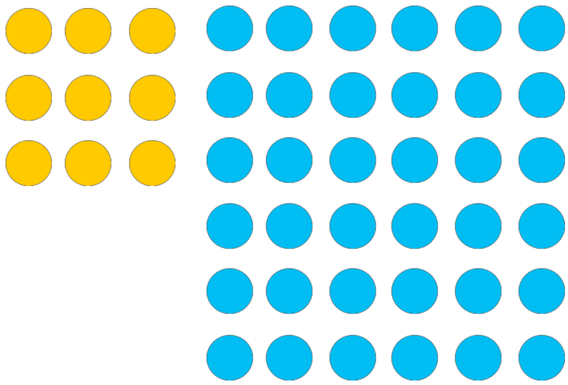




30 x 7 = 3 x (7 x 10)

Looking at the structure contained within the representation

Noticing and making connections



4 lots of 9 has the same value as 36

4 lots of 9 has the same value as 6 lots of 6

36 is equal to 4 lots of 3^2

6^2 is equal to 4×3^2

$3^2 \times 4 = 36$

A quarter of 36 is 9

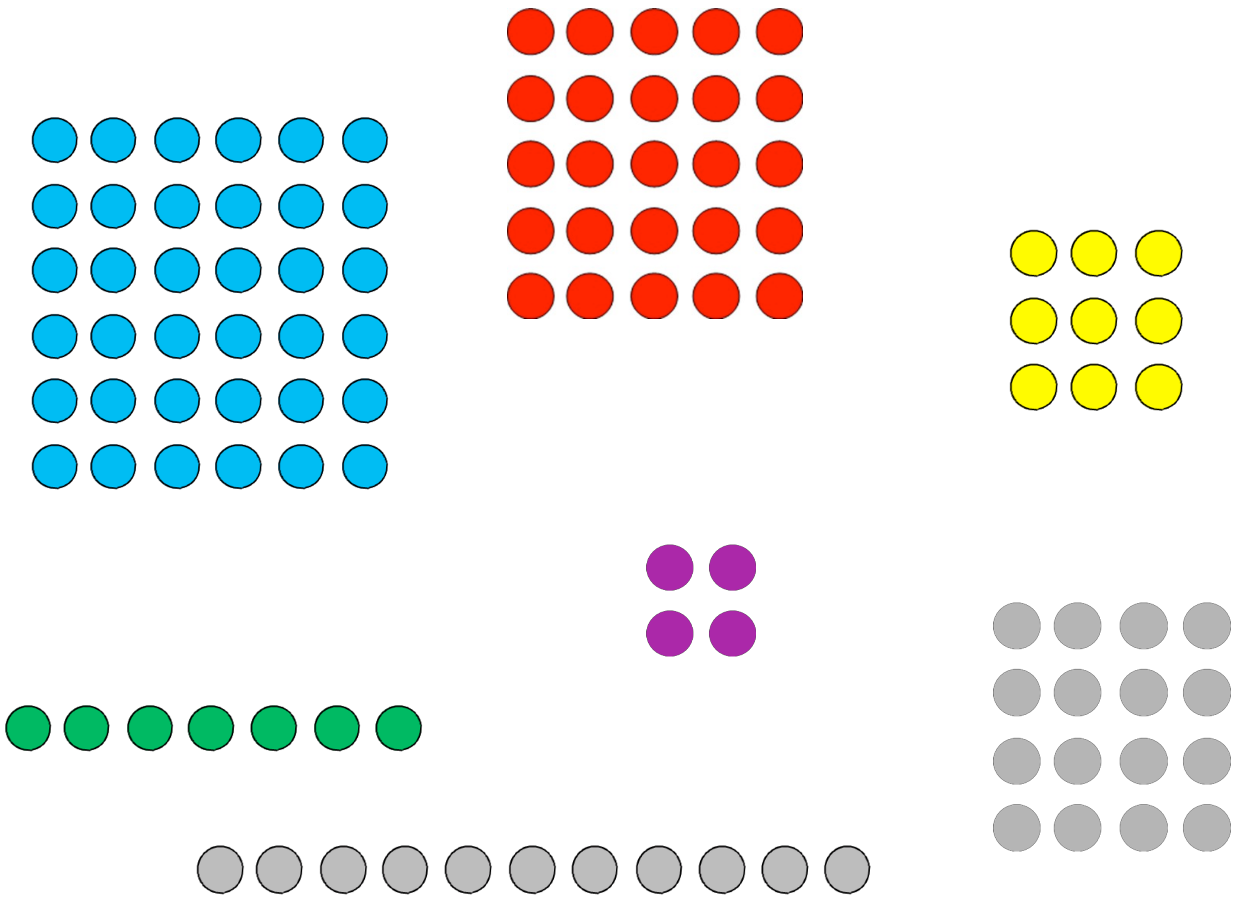
$3^2 = 36 \div 4$

$6^2 = 3^2 \times 4$

$6^2 \div 4 = 3^2$

$\frac{3}{4}$ of $36 = 3 \times 3^2 = 3^3$

Make connections



Designing questions

Noticing and making connections

8 x 9
9 x 6
17 x 5
15 x 7
37 x 4

This is an example of a random set of questions

When we ask children (or adults) where they started, they will usually say that they started with the first question, then 'did' the second question and so on until all have been completed. There is nothing to notice! The focus is on whether the product is right or wrong.

Now look at the set of questions below and ask children what they notice.

What do you notice?

Stick this sticker into the middle of you page and annotate it to show what you have noticed.

8 x 7
13 x 7
16 x 7
4 x 7
32 x 7
2 x 4 x 7

Here are some of the things we want children to notice and talk about:

The multiplier is the same.

One calculation involves three numbers and two multiplication signs - (the associative property of multiplication).

Which calculation they will do first and why.

Which order they will do the calculations and why.

What other calculations you would add to this set and why.

Which calculation you did last and why...

Sets of questions to develop reasoning about multiplication

14 x 4 14 x 2 7 x 8	12 x 3 12 x 6 24 x 6	7 x 5 14 x 5 3.5 x 10
9 x 8 18 x 8 18 x 16	16 x 3 4 x 3 8 x 6	8 x 9 24 x 3 12 x 3
8 x 0 8 x 8 4 x 4	6 x $\frac{1}{2}$ 6 x 2 6 x 1	6 x 12 12 x 12 6 x 24

Multiplication

Formal written methods

"Algorithms - a structured series of procedures that can be used across problems, regardless of the numbers - do have an important place in mathematics. After students have a deep understanding of number relationships and operations and have developed a repertoire of computation strategies …

"Algorithms should not be the primary goal of computation instruction… Using algorithms, the same set of steps with all problems, is antithetical to calculating with number sense. Calculating with number sense means that one should look at the numbers first and then decide on a strategy that is fitting - and efficient. Developing number sense takes time; algorithms taught too early work against the development of good number sense. Children who learn to think, rather than to apply the same procedures by rote regardless of the numbers, will be empowered. They will not see mathematics as a dogmatic, dead discipline, but as a living, creative one. They will thrive on inventing their own rules, because these rules will serve afterwards as the foundation for solving other problems."

Fosnot. C.T. and Dolk. M. (2001) Young mathematicians at work: *Constructing Multiplication and Division. Heinemann p.102.*

Short multiplication

$$12 \times 7 = 84$$

factor × factor = product

Short multiplication

Two-digit x one-digit

No carrying

$$\begin{array}{r} 22 \\ \times \quad 4 \\ \hline 88 \\ \hline \end{array}$$

Short multiplication

Carrying units

$$\begin{array}{r} 23 \\ \times \quad 4 \\ \hline 92 \\ {\scriptstyle 1} \end{array}$$

Short multiplication

Carrying from tens into a new place - hundreds

$$\begin{array}{r} 32 \\ \times \quad 4 \\ \hline 128 \end{array}$$

Short multiplication

Carrying from units to tens and from tens to a new place - hundreds

$$\begin{array}{r} 33 \\ \times \quad 4 \\ \hline 132 \\ {\scriptstyle 1} \end{array}$$

$$\begin{array}{r} 38 \\ \times \quad 7 \\ \hline 266 \\ {\scriptstyle 5} \end{array}$$

Short multiplication

$$12 \times 7 = 84$$

factor × factor = product

Short multiplication

Three-digit x one-digit

$$\begin{array}{r} 116 \\ \times\quad 3 \\ \hline 348 \\ \hline {}_{1} \end{array}$$

Also include calculations which involve using numbers with the same three digits - for example

116 x 3

161 x 3

611 x 3

Numbers with a zero

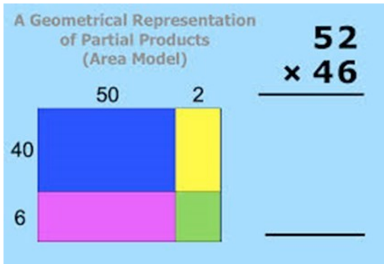
$$\begin{array}{r} 307 \\ \times\quad 4 \\ \hline 1228 \\ \hline {}_{2} \end{array}$$

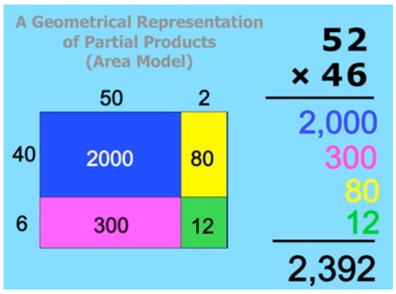
$$\begin{array}{r} 370 \\ \times\quad 4 \\ \hline 1480 \\ \hline {}_{2} \end{array}$$

Expanded vertical method leading to long multiplication.

Make links with the grid method

Two-digit x two-digit





Long multiplication

Two-digit x two-digit

$$\begin{array}{r} 43 \\ \times\quad 27 \\ \hline 860 \\ 301 \\ \hline {}_{2} \\ 1161 \\ \hline \end{array}$$

12 x 7 = 84

factor x factor = product

Long multiplication

Two-digit x two-digit

$$
\begin{array}{r}
43 \\
\times \quad 27 \\
\hline
860 \\
301 \\
\hline
{\scriptstyle 2} \\
1161 \\
\hline
\end{array}
$$

Children need to understand and be able to explain where this zero comes from.

Long multiplication

Three-digit x two-digit

$$
\begin{array}{r}
124 \\
\times \quad 27 \\
\hline
2480 \\
868 \\
\hline
{\scriptstyle 1\ 2} \\
3348 \\
\hline
{\scriptstyle 1\ 1}
\end{array}
$$

Children need to understand and be able to explain where this zero comes from.